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EFFECT OF FRICTION IN A CONTACT PROBLEM FOR A PLATE WITH A PIN

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A solution is obtained for a contact problem concerning the tension of a rectangular elastic plate with a circular hole into which a rigid stationary pin has been inserted. There is a small gap between the hole and the pin, which is of circular cross section. Friction acts in the contact region in accordance with the Coulomb law. The finite-element method and the Boussinesq principle are used to determine the load that realizes a specified contact region. Two variants of boundary conditions on the contour of the hole are examined.

1. Equations of the Problem. The expressions for the strains in terms of the displacements, the relations of the Hooke law, and the equilibrium equations for a plane stress state in the Cartesian coordinates x_1, x_2 [1, 2] are taken in the form

$$e_{11} = u_{1,1}, \quad e_{22} = u_{2,2}, \quad 2e_{12} = u_{1,2} + u_{2,1}, \quad e_{11} = E^{-1}(\sigma_{11} - \nu\sigma_{22}),$$
$$e_{22} = E^{-1}(\sigma_{22} - \nu\sigma_{11}), \quad e_{12} = (1+\nu)E^{-1}\sigma_{12}, \quad \sigma_{11,1} + \sigma_{12,2} = 0, \quad \sigma_{12,1} + \sigma_{22,2} = 0. \tag{1.1}$$

Here E is the Young modulus, ν is the Poisson ratio, u_i are the displacements, e_{ij} are the strains, σ_{ij} are the stresses $(i, j = 1,\ 2)$, and the subscripts 1 and 2 after a comma denote partial differentiation with respect to x_1 and x_2, respectively. As in [2], without loss of generality we assume that the plate is of a unit thickness that remains constant.

2. Boundary Conditions. We are given a rectangular plate with a circular hole of the radius R. In light of the symmetry of the solution relative to the axis $x_2 = 0$, we examine only the upper half of the plate (Fig. 1). The following boundary conditions are assigned on its contour away from the edge of the hole:

$$\sigma_{11} = \sigma_{12} = 0 \quad \text{for} \quad x_1 = -L_1,\ 0 \leqslant x_2 \leqslant H,$$
$$u_1 = w, \quad u_2 = 0 \quad \text{for} \quad x_1 = L_2,\ 0 \leqslant x_2 \leqslant H, \tag{2.1}$$
$$u_2 = 0, \quad \sigma_{12} = 0 \quad \text{for} \quad \begin{cases} x_2 = H, & -L_1 \leqslant x_1 \leqslant L_2, \\ x_2 = 0, & -L_1 \leqslant x_1 \leqslant -R, \quad R \leqslant x_1 \leqslant L_2. \end{cases}$$

A perfectly rigid stationary pin is inserted in the hole. The cross section of the pin is a circle with the radius $R_1 = R - c$ $(c = \varepsilon R$, where $0 < \varepsilon \ll 1)$ and its center is at the point with the Cartesian coordinates $(-c, 0)$. For any point on the contour of the hole Γ, we have

$$\cos\theta = \rho^{-1}(\varepsilon + \cos\varphi), \quad \sin\theta = \rho^{-1}\sin\varphi, \quad \rho = (1 + \varepsilon^2 + 2\varepsilon\cos\varphi)^{1/2}, \tag{2.2}$$

where θ is the angle between the axis x_1 and a normal to the contour of the pin Γ_c at the point closest to the point being examined (R, φ) on Γ and (r, φ) is a polar coordinate system $(x_1 = r\cos\varphi,\ x_2 = r\sin\varphi)$. The components of the displacement u_ρ and u_θ and forces q_ρ and q_θ vectors on Γ, taken in projections on a normal and a tangent to Γ_c, are expressed via the components of the displacements u_r and u_φ and the stresses σ_{rr}, $\sigma_{\varphi\varphi}$, and $\sigma_{r\varphi}$ in the polar coordinate system by the formulas

$$u_\rho = u_r\cos\alpha - u_\varphi\sin\alpha, \quad u_\theta = u_r\sin\alpha + u_\varphi\cos\alpha, \tag{2.3}$$

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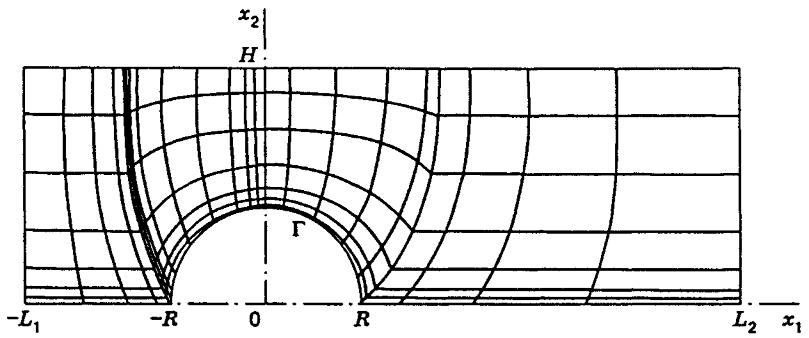

Fig. 1

$$q_\rho = \sigma_{rr}\cos\alpha - \sigma_{r\varphi}\sin\alpha, \quad q_\theta = \sigma_{rr}\sin\alpha + \sigma_{r\varphi}\cos\alpha, \quad \alpha = \varphi - \theta.$$

The condition expressing the fact that the edge of the hole impenetrates within the contour of the pin is represented in the form [2]

$$(\rho R + u_\rho)^2 + u_\theta^2 = (R + c\ \cos\varphi + u_r)^2 + (u_\varphi - c\ \sin\varphi)^2 \geqslant R_1^2 \tag{2.4}$$

or

$$u_\rho + (2\rho R)^{-1}(u_\rho^2 + u_\theta^2) \geqslant u_{\rho c} \quad [u_{\rho c} = -c\rho^{-1}(1 + \cos\varphi)]. \tag{2.5}$$

Linearizing this condition, we obtain the inequality

$$u_\rho \geqslant u_{\rho c}. \tag{2.6}$$

This inequality is stronger than (2.5) in the sense that displacements which satisfy (2.6) also satisfy (2.5).

We shall formulate two variants of boundary conditions on Γ.

Boundary Conditions (*a*). We assume that $u_\rho = u_{\rho c}$ in the contact region Γ_1 and that the edge of the hole is pressed against the pin. Thus, $q_\rho < 0$. Let friction act on Γ_1 in accordance with the Coulomb law [3]. Then the permissible values of the forces are bounded by the inequalities $q_\rho < 0$ and $|q_\theta| \leqslant \mu|q_\rho|$ or $f_1 = \mu q_\rho + q_\theta \leqslant 0$, $f_2 = \mu q_\rho - q_\theta \leqslant 0$, and $F = f_1 f_2 \geqslant 0$ (μ is the friction coefficient). Thus, these values occupy an angle bounded by the straight lines $f_1 = 0$ and $f_2 = 0$ in the half-plane $q_\rho < 0$ in Cartesian coordinates q_ρ, q_θ. The minimum possible angles of inclination of the vector of the forces (q_ρ, q_θ) to Γ_c are given by the conditions $f_1 = 0$ or $f_2 = 0$. The modulus of this vector is unrestricted by the friction law.

Sliding of the pin with friction against the edge of the hole at a nonzero velocity $\dot{u}_\theta$ (the dot denotes partial differentiation with respect to the loading parameter of the plate, which we shall refer to as the time τ) can occur only when the values of the forces are located on one of the boundary lines and remain on this line during additional loading of the plate $f_1 = \dot{f}_1 = 0$ or $f_2 = \dot{f}_2 = 0$. Energy dissipation may be nonnegative ($Q = q_\theta \dot{u}_\theta \geqslant 0$), while the frictional force ($-q_\theta$) acts in the direction opposite to the velocity $\dot{u}_\theta$. The absolute values of $\dot{u}_\theta$ during the sliding of the pin are independent of the values of q_ρ and q_θ. The nonslip $\dot{u}_\theta = 0$ is realized at the remaining points of the contact region, where the slip conditions are not satisfied.

The conditions $u_\rho \geqslant u_{\rho c}$, $\sigma_{rr} = \sigma_{r\varphi} = 0$, and $q_\rho = q_\theta = 0$ are satisfied on the free part of the edge of the hole Γ_2.

Thus, the following boundary conditions are imposed on the contour of the hole:

$$\begin{gathered} u_\rho = u_{\rho c}, \quad \dot{u}_\theta = 0, \quad q_\rho < 0, \quad F > 0 \quad \text{on } \Gamma_1', \\ u_\rho = u_{\rho c}, \quad f = 0, \quad q_\rho < 0, \quad \begin{cases} \dot{u}_\theta = 0, & \text{if } \dot{f} < 0, \\ Q \geqslant 0, & \text{if } \dot{f} = 0, \end{cases} \quad \text{on } \Gamma_1'', \\ q_\rho = q_\theta = 0, \quad u_\rho \geqslant u_{\rho c} \quad \text{on } \Gamma_2. \end{gathered} \tag{2.7}$$

At each point Γ_1'', as f we take a function f_1 or f_2 that is zero at this point; $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 = \Gamma_1' \cup \Gamma_1''$.

The inequality $Q \geqslant 0$ can be replaced by $\dot{u}_\theta \geqslant 0$ when $f = f_1 = 0$ and $q_\theta = -\mu\, q_\rho > 0$ and by $\dot{u}_\theta \leqslant 0$ when $f = f_2 = 0$ and $q_\theta = \mu q_\rho < 0$. With allowance for $q_\rho < 0$, the condition $F > 0$ on Γ_1' is equivalent to the inequalities $f_1 < 0$ and $f_2 < 0$, which are linear in the sought functions.

The partitions $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 = \Gamma_1' \cup \Gamma_1''$ are completely determined by the forces q_ρ and q_θ at the current moment of time. The change in the external load acting on the plate at this moment affects only the partitioning of Γ_1'' into parts in which $\dot{f} < 0$ and $\dot{f} = 0$. The regions Γ_1', Γ_1'', and Γ_2, the form of the function f (f_1 or f_2) on Γ_1'', and the forces q_ρ and q_θ on Γ_1 depend on the loading history of the plate and are found from the solution of the problem.

If we insert the expressions for u_ρ, u_θ, q_ρ, and q_θ from (2.3) and the expression for $u_{\rho c}$ from (2.5) into (2.7) then with allowance for (2.2) the radical ρ is excluded from (2.7) and the coefficients at u_r, u_φ, σ_{rr}, and $\sigma_{r\varphi}$ contain the parameter ε. Thus, the solution obtained with the use of (2.7) may depend nonlinearly on ε.

Boundary Conditions (*b*). We approximate the nonpenetration condition (2.4) by means of the inequality $u_r \geqslant u_{rc}$, where $u_{rc} = -c\,(1 + \cos\varphi)$. This inequality is stronger than (2.4). Then the boundary conditions are formulated similarly to (2.7), with the replacement of u_ρ, u_θ, q_ρ, q_θ, and $u_{\rho c}$ by u_r, u_φ, σ_{rr}, $\sigma_{r\varphi}$, and u_{rc}, respectively. Now, in contrast to (2.7), the parameters of the gap c and ε appear in the boundary conditions only in the expression for u_{rc} and enter linearly into the terms that are not dependent on the sought functions.

Let us introduce the notation for the normal and tangential components of the displacements and the forces. The notation is the same for both variants of boundary conditions on Γ, taking the form $u = u_\rho$, $v = u_\theta$, $p = q_\rho$, $q = q_\theta$, and $u_c = u_{\rho c}$ for conditions (a) and $u = u_r$, $v = u_\varphi$, $p = \sigma_{rr}$, $q = \sigma_{r\varphi}$, and $u_c = u_{rc}$ for conditions (b). Then the boundary conditions in both variants are represented in the form

$$
\begin{gathered}
u = u_c, \quad \dot{v} = 0, \quad p < 0, \quad F > 0 \quad \text{on } \Gamma_1', \\
u = u_c, \quad f = 0, \quad p < 0, \quad \begin{cases} \dot{v} = 0, & \text{if } \dot{f} < 0, \\ Q \geqslant 0, & \text{if } \dot{f} = 0, \end{cases} \quad \text{on } \Gamma_1'', \\
p = q = 0, \quad u \geqslant u_c \quad \text{on } \Gamma_2.
\end{gathered}
\tag{2.8}
$$

Here $f_1 = \mu\, p + q$, $f_2 = \mu\, p - q$, $Q = q\, \dot{v}$, and the regions Γ_1', Γ_1'', and Γ_2 and the function f (f_1 or f_2) on Γ_1'' may be different for conditions (a) and (b). In variant (a), the boundary conditions (2.8) coincide with (2.7).

The rate of energy dissipation $\dot{\Phi}_f$, the power $\dot{\Phi}$ of the tractive force P, and the rate of change in strain energy of the plate $\dot{\Phi}_e$ are connected by the relations

$$
\dot{\Phi}_f = \int\limits_{\Gamma_1''} Q\, d\Gamma = \dot{\Phi} - \dot{\Phi}_e, \quad \dot{\Phi} = P\, \dot{w}, \quad P = \int\limits_0^H \sigma_{11}\, dx_2,
$$

where P is calculated for $x_1 = L_2$. Integrating over τ, we find that $\Phi = \Phi_f + \Phi_e$,

$$
\Phi_e = \int\limits_\Omega \frac{E}{2(1-\nu^2)} \left[e_{11}^2 + 2\nu e_{11} e_{22} + e_{22}^2 + 2(1-\nu) e_{12}^2\right] dx_1\, dx_2.
$$

The work done by the force P is expended to the deformation energy of the plate Φ_e and the energy Φ_f dissipated due to friction.

Thus, in the presence of friction and a gap, we have two contact problems for Eqs. (1.1) with boundary conditions (2.1) and (2.8) in variants (a) and (b). At each moment of time, their solution depends on the loading history of the plate. The methods used to solve the problems are similar in both variants.

In the case where there is no gap $\varepsilon = 0$ and the displacement w increases monotonically from zero in the initial undeformed state of the plate, we can use (2.8) to obtain the same boundary conditions for both variants

$$
u = v = 0, \quad p < 0, \quad F > 0 \quad \text{on } \Gamma_1',
$$

$$u = 0, \quad f = 0, \quad p < 0, \quad Q_1 \geqslant 0 \quad \text{on} \quad \Gamma_1'', \tag{2.9}$$
$$p = q = 0, \quad u \geqslant 0 \quad \text{on} \quad \Gamma_2.$$

The solution of the problem for Eqs. (1.1) with boundary conditions (2.1) and (2.9) is linearly proportional to w. The regions Γ_1', Γ_1'', and Γ_2 are formed at the initial moment of time immediately after the displacement w is assigned as small a value as desired, and they remain unchanged with an increase in the displacement. Each equality and inequality in (2.9) remains valid for any $w > 0$. Since $f = \dot{f} = 0$, for any w the function f also remains the same at each point on Γ_1'': f_1 or f_2. With allowance for the linear dependence of q and v on w, the condition $Q \geqslant 0$ in (2.8) is replaced by a condition stipulating that the energy dissipation is nonnegative ($Q_1 = 0.5qv \geqslant 0$).

The boundary conditions in the limiting problems (in the absence of friction) [2] follow from (2.8) and (2.9) as $\mu \to 0$.

The contact problem for a plate with a pin was solved in [2, 4, 5] without allowance for friction and in [6] with allowance for Coulomb friction. In the second case, the difference in the displacements of the edge of the hole and the contour of the elastic pin was, however, assigned without allowance for the history of their interaction.

3. Formulation of Equilibrium Problems with a Specified Contact Region. We introduce a new variable $\eta = 1 - \varphi/\pi$ $(0 \leqslant \eta \leqslant 1)$ on the contour of the hole Γ. This variable increases along Γ in the left-to-right direction from $\eta = 0$ at the point $(-R, 0)$ to $\eta = 1$ at the point $(R, 0)$ (Fig. 1).

Then, examining the loading of the plate by means of a monotonically increasing displacement of its right side w, we assume that in the algorithm being developed the regions Γ_1', Γ_1'', and Γ_2 on Γ occupy the respective segments $0 \leqslant \eta \leqslant b$, $b < \eta \leqslant l$, and $l < \eta \leqslant 1$ with the end points $\eta = b$ and $\eta = l$. Thus, the contact region $\Gamma_1 = \Gamma_1' \cup \Gamma_1''$ is the segment $0 \leqslant \eta \leqslant l$. At all the points of Γ_1'', the frictional forces $(-q)$ are assumed to act along Γ_1'' in the same direction. Thus, we assume that the function f (f_1 or f_2) is equal to zero. The values of b and l may depend on the displacement w, the gap c, the friction coefficient μ, and the boundary conditions (2.8) or (2.9).

The solution of the problem for Eqs. (1.1) with boundary conditions (2.1) and (2.8) is found with the length l of the contact region which increases monotonically in steps. The value of l at the moment $(\tau + \Delta\tau)$ is assigned at the end of each step from τ to $(\tau + \Delta\tau)$. The rates of shear displacements on Γ at the end of a step are determined from the formula $\dot{v} = (v - v_\tau)/\Delta\tau$ (the values at the beginning of the step are denoted by the subscript τ, with no notation used to indicate the end of the step). We take $l = 0$ and $v = 0$ for $\tau = 0$ as the initial conditions for the first step.

Having discarded the inequalities in (2.8), we obtain the following boundary conditions:

$$u = u_c, \quad v = v_\tau \quad \text{for} \quad 0 \leqslant \eta \leqslant b,$$
$$u = u_c, \quad f = 0 \quad \text{for} \quad b < \eta \leqslant l, \tag{3.1}$$
$$p = q = 0 \quad \text{for} \quad l < \eta \leqslant 1.$$

Here the same function f (f_1 or f_2) is taken at all the points of the segment $b < \eta \leqslant l$. The solution of the problem for Eqs. (1.1) with boundary conditions (2.1)–(3.1) (henceforth referred to as problem A) is used to determine the state of equilibrium of the plate at the end of a step with a specified contact region.

Now we introduce two more problems A1 and A2, these problems differing from A only in the fact that the following are assigned in the boundary conditions:

— $u = v = 0$ in A1 and $u = c^{-1}u_c$ and $v = c^{-1}v_\tau$ in A2 for $0 \leqslant \eta \leqslant b$,

— $u = 0$ and $f = 0$ in A1 and $u = c^{-1}u_c$ and $f = 0$ in A2 for $b < \eta \leqslant l$,

— $u_1 = 1$ and $u_2 = 0$ in A1 and $u_1 = 0$ and $u_2 = 0$ in A2 for $x_1 = L_2$, $0 \leqslant x_2 \leqslant H$.

Without the inequalities, the boundary conditions (2.9) take the form

$$u = v = 0 \quad \text{for} \quad 0 \leqslant \eta \leqslant b,$$
$$u = 0, \quad f = 0 \quad \text{for} \quad b < \eta \leqslant l, \tag{3.2}$$

$$p = q = 0 \quad \text{for} \quad l < \eta \leqslant 1.$$

The problem for Eqs. (1.1) with boundary conditions (2.1) and (3.2) has a solution that is linearly proportional to w. It coincides with problem A1 for $\varepsilon = 0$ and $w = 1$, the values of b and l are identical in both problems, and the same function f (f_1 or f_2) is equal to zero on the segment $b < \eta \leqslant l$. We also seek values of b, l, and f at which restrictions in the form of inequalities in (2.9) are satisfied.

4. Solution of Problems. The following algorithm is used at each moment of time during loading of the plate to find the solution. The plate is divided into Lagrangian finite elements (tetragonal, nine-node, and isoparametric [7]) (Fig. 1). Each subdivision contains about 160 finite elements and 1300 sought variables, i.e., the components of the displacements of the nodes of the elements.

Using the principle of virtual displacements, we formulate the systems of finite-element equations of problems A, A1, and A2. These systems have the same asymmetric matrix of coefficients of the sought variables and differ from one another only in their right sides, which are determined by the displacements assigned by the boundary conditions on the contour of the plate. The Gauss three-point quadrature formula is used to calculate the integrals over an area of an element in the process of integration over each local coordinate. The systems of equations that are formulated are solved by the Gauss elimination method [7, 8] with allowance for the band nature of the coefficient matrix and the symmetry of most of this matrix relative to the principal diagonal. The matrix is calculated and reduced to a triangular form only once for both problems A1 and A2.

The hoop stresses $\sigma_{\varphi\varphi}$ at the nodes on Γ are found by interpolation of the stresses calculated at the points of integration over the area of the elements. The normal and shearing forces p and q are determined through the generalized forces acting at the nodes in accordance with the principle of virtual displacements.

In the finite-element formulation, we seek the solution of problem A as the sum of the solutions of problems A1 and A2 with the coefficients w and c. This sum has the form

$$\mathbf{U} = w\mathbf{U}^{(1)} + c\mathbf{U}^{(2)}. \tag{4.1}$$

Here and below, $\mathbf{U}$ denotes the global vectors of the sought variables, i.e., the components of the displacements of the nodes of the elements; the superscripts 1 and 2 in parentheses denote problem A1 and problem A2, respectively.

In accordance with (4.1), we have

$$f_b = w\, f_b^{(1)} + c\, f_b^{(2)}, \quad p_l = w\, p_l^{(1)} + c\, p_l^{(2)}, \tag{4.2}$$

where the subscripts b and l denote the values of f and p at the points $\eta = b$ and $\eta = l$, respectively; we take the function f (f_1 or f_2) that is equated to zero in (3.1) on the segment $b < \eta \leqslant l$.

In accordance with the Boussinesq principle [3, 9], we assume that $p_l = 0$. We find from (4.2) that

$$w = -c\, p_l^{(2)} /\, p_l^{(1)}. \tag{4.3}$$

Inserting (4.3) into (4.1), we resort to iteration to find the position of the point $\eta = b$ at which $f_b = 0$. If the function f (f_1 or f_2) is chosen correctly, we obtain $\mathbf{U}$ as the solution of the problem for Eqs. (1.1) with boundary conditions (2.1) and (2.8). The restrictions in the form of the inequalities in (2.8) are satisfied, and w is the displacement that realizes the specified contact region. On the segment $b < \eta \leqslant l$ we have $f = 0$, $f_\tau \leqslant 0$, and approximately $\dot{f} = (f - f_\tau)/\Delta\tau \geqslant 0$, and therefore we should check only that Q is nonnegative. The case $\dot{f} < 0$ is not examined here. The transition from slip at the beginning of a step at the moment τ to nonslip at the end of the step, with satisfaction of the inequality $\dot{f} < 0$, can occur at points of the segment from $\eta = b_\tau$ to $\eta = b$ for $b_\tau < b$. It is possible to proceed to the next time step after the state of equilibrium of the plate at the end of the current step has been determined.

We note that in the finite-element formulation the boundary conditions are applied to a discrete set of nodes which includes the points $\eta = b$ and $\eta = l$. The nonslip conditions $u = u_c$ and $v = v_\tau$ are assigned at the nodes for $0 \leqslant \eta \leqslant b$, the slip conditions $u = u_c$ and $f = 0$ are assigned for $b < \eta \leqslant l$, and the free-edge conditions $p = q = 0$ are assigned for $l < \eta \leqslant 1$. Satisfaction of the equations $f_b = p_l = 0$ in accordance with the Boussinesq principle leads to satisfaction of the nonslip and slip conditions at the point $\eta = b$, and the

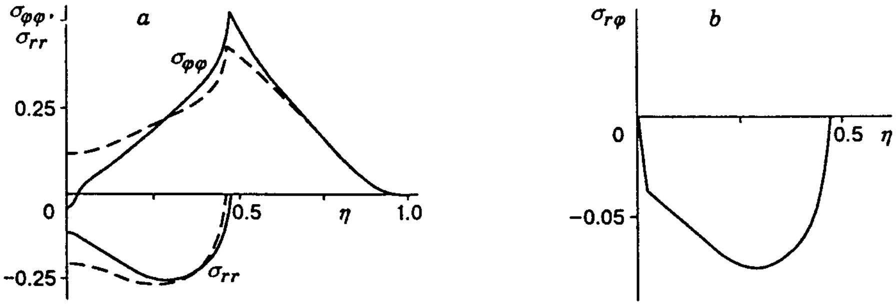

Fig. 2

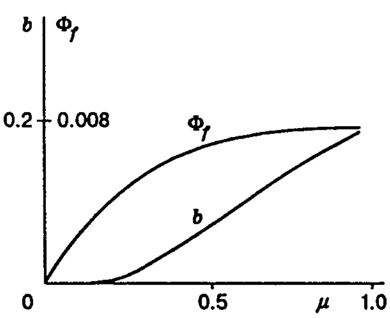

Fig. 3

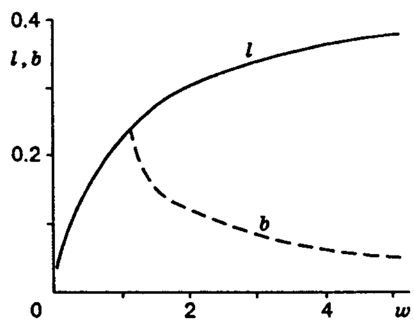

Fig. 4

slip and free-edge conditions at the point $\eta = l$. In this sense, there is a continuous transition along Γ from one set of boundary conditions to the other.

In variant (b), the vectors $\mathbf{U}^{(b1)}$ and $\mathbf{U}^{(b2)}$ are independent of c and ε. As follows from (4.1) and (4.3), the solution of the problem $\mathbf{U}^{(b)}$ is linearly proportional to the gap c. In variant (a), the coefficients of the finite-element equations of problems A1 and A2 and the vectors $\mathbf{U}^{(a1)}$ and $\mathbf{U}^{(a2)}$ depend on ε. Thus, the solution $\mathbf{U}^{(a)}$ can depend nonlinearly on ε. The superscripts a and b in parentheses denote boundary-condition variants (a) and (b), respectively. Problems A1 and A2 and the contact problem as a whole in variant (a) are solved anew with each new value of c.

We shall solve separately problem A1 for variant (b). We choose the appropriate function f (f_1 or f_2), equal to zero on the segment $b < \eta \leqslant l$, and we use iteration to find values of $l = l_*$ and $b = b_*$ such that $p_l^{(b1)} = f_b^{(b1)} = 0$. Then, in accordance with the Boussinesq principle, we obtain $\mathbf{U}^{(b1)} = \mathbf{U}_*^{(b1)}$ as the solution of the problem for Eqs. (1.1) with boundary conditions (2.1) and (2.9) for $w = 1$.

5. Computation Results. We pass over to dimensionless quantities by multiplying x_1, x_2, and r by the nondimensionalizing factor R^{-1}, the displacements and the gap c by L_0^{-1}, the strains by $\omega = RL_0^{-1}$, the stresses by ωE^{-1}, and the energies Φ_e and Φ_f and the work Φ by $E^{-1}L_0^{-2}$ (L_0 is a constant having the dimension of length). We keep the previous notation for the nondimensionalized quantities. Now, we have $R = 1$, $H = L_1 = 2.5$, $L_2 = 5$, and $c = \omega\varepsilon$. The Poisson ratio is $\nu = 0.3$.

In the equations and boundary conditions that have been changed to dimensionless form, c and ε are taken as the dimensionless parameters associated with the gap in variant (a), while only c is taken as such a parameter in variant (b). In both variants, we can use the prescribed values of c and ε to determine $\omega = c\varepsilon^{-1}$ and change back from the dimensionless forms of the sought functions to their dimensional forms.

In the absence of a gap, we have $c = \varepsilon = 0$. Since the solution is linearly proportional to w, we assume that $w = 1$. Returning to the dimensional quantities and assigning values to w and R, we find that $L_0 = w$ and $\omega = RL_0^{-1}$.

Let us examine the solution of the contact problem with friction in the case of the absence of a gap,

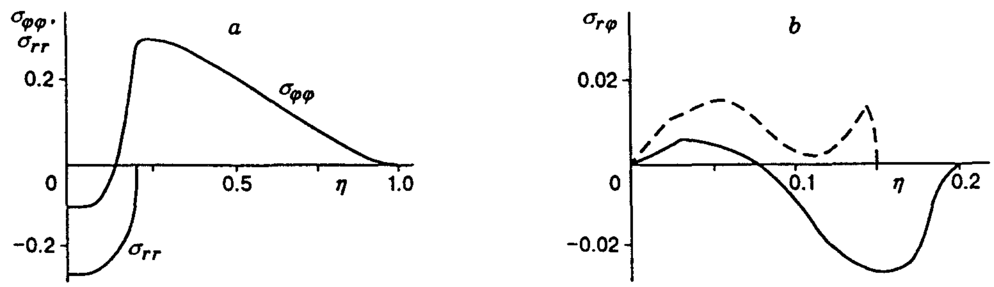

Fig. 5

TABLE 1

Solution	μ	l	b	P	Φ_e	$\sigma^*_{\varphi\varphi}$
$U^{(b)}$	0.3	0.15	0.1435	0.07831	0.01797	0.1641
$U^{(a)}$	0.3	0.1452	0.1422	0.07775	0.01785	0.1628
$U^{(b)}$	0	0.1511	0	0.07835	0.01796	0.1664

obtained for Eqs. (1.1) with boundary conditions (2.1) and (2.9) for $\varepsilon = 0$, $\mu = 0.3$, $w = 1$, $P = 0.2472$, $l = l_* = 0.4755$, and $b = b_* = 0.01887$ (the angles πl and πb are equal to 85.3 and 3.2°, respectively). The subdivision of the plate into finite elements is shown in Fig. 1. We have 7 elements across the width of the plate and 22 elements along the length, with 154 elements in all.

The solid curves in Fig. 2 show the stresses σ_{rr}, $\sigma_{\varphi\varphi}$, and $\sigma_{r\varphi}$ on Γ as a function of η. The sections of the dashed curves in Fig. 2a that deviate from the solid curves show the values of σ_{rr} and $\sigma_{\varphi\varphi}$ in the absence of friction ($\varepsilon = 0$, $\mu = 0$, $w = 1$, $P = 0.2317$ and $l = 0.4619$). The distributions of σ_{rr} are not cosine curves in either problem (with and without friction). We have $\sigma_{rr} = \sigma_{r\varphi} = 0$ for $l \leqslant \eta \leqslant 1$. The maxima $\sigma_{\varphi\varphi} = \sigma^*_{\varphi\varphi}$ are reached on the free part of Γ near the points $\eta = l$, while the maximum absolute values of σ_{rr} are reached at internal points of the contact region. Due to friction around the point $\eta = 0$, there is a small region where $\sigma_{rr} < 0$ and $\sigma_{\varphi\varphi} < 0$. The value of $\sigma^*_{\varphi\varphi}$ and the stress concentration factor $k = HP^{-1}\sigma^*_{\varphi\varphi}$ increase from $\sigma^*_{\varphi\varphi} = 0.4280$ and $k = 4.615$ for $\mu = 0$ to $\sigma^*_{\varphi\varphi} = 0.5281$, and $k = 5.338$ for $\mu = 0.3$. The maximum absolute values of σ_{rr} decrease. The energy $\Phi_f = 0.005288$ is low compared to the strain energy in the plate $\Phi_e = 0.1183$ and amounts to 4.5%. In the absence of friction, we have $\Phi = \Phi_e = 0.1159$.

In the absence of a gap, the contact and nonslip regions become longer with increase in μ for $\mu \leqslant 1$ in the solution of the contact problem with friction. Here b is negligibly small as long as $\mu < 0.2$ (Fig. 3). The values of l, P, and $\sigma^*_{\varphi\varphi}$ increase almost linearly in relation to μ. We have $l = 0.5027$ and $b = 0.1893$ for $\mu = 1$. The energy Φ_f increases more slowly with increase in μ.

The solution $U^{(b)}$ of the problem for Eqs. (1.1) with boundary conditions (2.1) and (2.8) in variant (b) was calculated upon variation of l from 0.03 to 0.45 in 31 steps for $c = 1$ and $\mu = 0.3$. We assumed zero shear displacements in the nonslip region for the initial value of $l = 0.03$.

Figure 4 shows the functions l and b versus w. The slip regions are small for $l \leqslant 0.24$, and the diagrams of l and b nearly merge and are represented by a solid curve. When the length of the slip region is too small, we assume that $l = b$. We used iteration to successively find the values of $l = b = 0.1910$ and 0.1997 such that $\sigma_{rr} = \sigma_{r\varphi} = 0$ at the point $\eta = l = b$. In the slip regions, we have $f_1 = 0$, $\dot{u}_\varphi > 0$, and $\sigma_{r\varphi} > 0$ for $l < 0.1910$ and $f_2 = 0$, $\dot{u}_\varphi < 0$, and $\sigma_{r\varphi} < 0$ for $l > 0.1997$. Beginning with $l = 0.24$, the length of the nonslip region b decreases with increase in l. The energy Φ_f is negligibly low as long as $l < 0.24$.

The solid curves in Fig. 5 show the values of σ_{rr}, $\sigma_{\varphi\varphi}$, and $\sigma_{r\varphi}$ on Γ for $l = b = 0.1997$, $w = 0.8045$, and $P = 0.1392$. The dashed curve corresponds to $l = 0.15$. The values of $\sigma_{r\varphi}$ in the contact region change from positive to negative during loading of the plate (Fig. 5b).

The difference between the solutions $U^{(a)}$ and $U^{(b)}$ of the problem for Eqs. (1.1) with boundary conditions (2.1) and (2.8) in variants (a) and (b) is small when calculated for the same value of w. When w is small, these solutions also differ negligibly from the solutions of the contact problems in variants (a) and (b) with the same gap in the absence of friction. Table 1 shows values of l, b, P, Φ_e, and $\sigma^*_{\varphi\varphi}$ calculated for $c = 1$, $\varepsilon = 0.05$, and $w = 0.4883$ on the basis of the solutions $\mathbf{U}^{(a)}$ and $\mathbf{U}^{(b)}$ in the presence and in the absence of friction ($\mu = 0$). The boundary conditions in variant (a) are less restrictive for the plate than the boundary conditions in variant (b): the values of l, b, P, Φ_e, and $\sigma^*_{\varphi\varphi}$ in $\mathbf{U}^{(a)}$ are lower than the corresponding values in $\mathbf{U}^{(b)}$.

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